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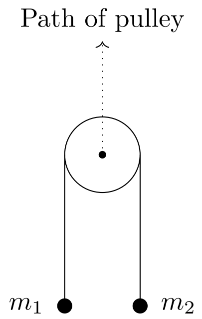
4801. Natural numbers a and b, where $a, b \geq 2$, are said to be *coprime* if they have no common factor other than 1. Prove that, if a and b are coprime, then $\log_a b$ is irrational.

4802. A pulley system consists of a light, inextensible string passed over a smooth pulley. Bobs of mass $m_1 < m_2$ are attached to the ends of the string. The vertical position of the pulley is adjustable. Initially, the system is at rest.

The bobs are released, and the pulley is given a constant vertical acceleration.



Determine, in terms of m_1, m_2 and g, the vertical accelerations of the pulley which will leave one of the masses at rest.

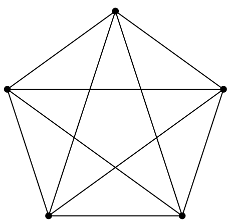
4803. Linear functions f, g are such that, for all $x \in \mathbb{R}$,

$$\mathrm{f}^3(x) = \mathrm{g}^3(x).$$

Show that f and g must be the same function.

4804. Sketch the graph $y = \displaystyle\sum_{r=0}^{6} \frac{1}{x - r + 3}$.

4805. When all of the diagonals of a regular pentagon are drawn, a new regular pentagon is formed at the centre.



Show that the ratio of lengths of these pentagons is $1 : 4\sin^2 18°$.

4806. Show that $\displaystyle\int_0^1 (\ln x)^3 \, dx = -6$.

4807. A cubic graph $y = \mathrm{f}(x)$ crosses the x axis at three distinct points, which are in AP. Show that this curve may be transformed, by a combination of stretches, translations and reflections, to the curve $y = x^3 - x$.

4808. In this question, the quadratic approximation to the cosine function is notated $\mathrm{c}_2(x)$. It is proposed that a quartic approximation to the cosine function be given by

$$\mathrm{c}_4(x) = 1 - \tfrac{1}{2}x^2 + kx^4.$$

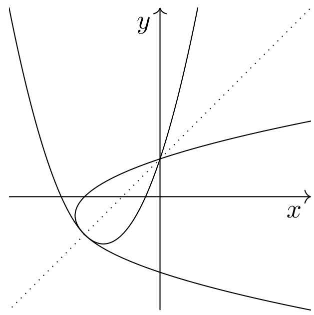
(a) Calculate a table of values, to 6sf, of $\cos(x)$ and $\mathrm{c}_2(x)$, for $x = 0.1, 0.2, 0.3$ radians.

(b) Show, by using logarithmic linearisation, that $\cos(x) - \mathrm{c}_2(x)$ is approximately quartic.

(c) Show that $k \approx \frac{1}{24}$.

4809. A particle moves with acceleration a and position $x \geq 0$ related by $a = \frac{3}{2}x^2$. Using the chain rule to write a in terms of v and x, show that, for large x, speed is approximately proportional to $x^{\frac{3}{2}}$.

4810. The parabola $y = ax^2 + bx + c$ is reflected in the line $y = x + c$.



Find the equation of the new parabola.

4811. This question concerns the closest approach of two adjacent branches of the curve $y = \tan x$.

(a) Show that the equation of the normal at the point $(p, \tan p)$ is $y = (p - x)\cos^2 p + \tan p$.

(b) Explain why a line of the form $y = m(x - \pi/2)$ represents the closest approach of two adjacent branches.

(c) Show that, to 4sf, the closest approach is a distance of 2.375.

4812. Prove that the three angle bisectors of a triangle ABC are concurrent.

4813. Determine the following integral:

$$\int \frac{1}{x^2 + 8x + 17} \, dx.$$

4814. Prove that, if $y = \mathrm{g}'(x)$ has rotational symmetry around point (a, b), then $y = \mathrm{g}(a + x) - \mathrm{g}(a - x)$ is linear, with gradient $2b$.

4815. The *insphere* of a polyhedron is the largest sphere that will fit inside it. Show that the insphere of a regular tetrahedron of side length 1 has radius $r = 1/\sqrt{24}$.

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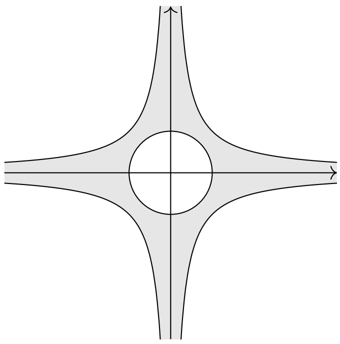
4816. (a) Prove the following *sum-to-product formula*:

$$\cos Q - \cos P \equiv 2\sin\left(\tfrac{P+Q}{2}\right)\sin\left(\tfrac{P-Q}{2}\right).$$

(b) For $0 \le x \le \frac{\pi}{2}$, solve the equation

$$\cos 6x + \sin 4x = \cos 2x.$$

4817. A region defined by an inequality is shown below. Points satisfying the inequality are shaded.



One of the following is the defining inequality. By ruling the other three out, determine which one:

① $(xy-1)(x^2+y^2-1) \le 0$.
② $(xy-1)(1-x^2-y^2) \le 0$.
③ $(x^2y^2-1)(x^2+y^2-1) \le 0$.
④ $(x^2y^2-1)(1-x^2-y^2) \le 0$.

4818. Data-generating processes C_n are defined as:

- n fair coins are tossed,
- the number of heads showing is recorded.

Processes C_{2k-1} and C_{2k}, for $k \in \mathbb{N}$, are performed separately. Determine the probability that C_{2k} produces more heads than C_{2k-1}.

4819. The parabola $y = x^2$ has two normals, at $x = p$ and $x = p + 2$, which meet at $x = \frac{15}{2}$. Find p.

4820. The function f is defined by

$$\text{f}(x) = x^4 - 3x^2 + x + 1.$$

A tangent is drawn to the curve $y = \text{f}(x)$ at $x = a$. This tangent re-intersects the curve at two distinct points $x = \pm b$, where $a \ne b$.

(a) Let g be a linear function.
Explain why transforming the problem such that $y = \text{f}(x)$ is mapped to $y = \text{f}(x) + \text{g}(x)$ will not change the x coordinates of the two re-intersections.

(b) Hence, find the equation of the tangent line.

4821. *Heron's formula* states that the area of a triangle, whose sides have lengths a, b, c, is given in terms of the semiperimeter $s = \frac{1}{2}(a+b+c)$, by

$$A = \sqrt{s(s-a)(s-b)(s-c)}.$$

Prove the formula.

4822. The cubic graph $y = ax^3 + bx^2 + cx + d$ crosses the x axis at $x = p, q, r$. Find, with coefficients in terms of a, b, c, the equation of the monic quartic that passes through the origin and is stationary at $x = p, q, r$.

4823. Find $\displaystyle\int xe^x \cos x \, dx$.

4824. A projectile is bouncing, without any loss of speed in doing so, between smooth surfaces defined by

$$y = \tfrac{\sqrt{3}}{3}|x|.$$

Initially, it is projected horizontally with speed u, and travels on the trajectory

$$y = \frac{u^2}{2g} - \frac{gx^2}{2u^2}.$$

Show that its ensuing motion is periodic.

4825. Variables $y_1, y_2, y_3, \ldots$ are related as follows:

$$\frac{dy_r}{dy_{r-1}} = 1 - \frac{1}{r^2}, \quad \text{for } r = 2, 3, 4, \ldots$$

Show that $y_n = \dfrac{n+1}{2n} y_1 + c$.

4826. In a game of tic-tac-toe, two players alternately place ○ and ×. A player who places three in a row horizontally, vertically or diagonally, wins. In the game shown, three moves have been played:

×		
○	○	

This game continues, ending in a draw. Show that there are three possible grids. Here, the "grid" is the final position on the board, irrespective of the order in which the moves were played.

4827. Sketch the curve $y = \sqrt{1 - x^{-1.8}}$ on the largest real domain over which it may be defined.

4828. Three couples sit down at random around a round table. Show that the expectation of the number of couples sitting together is $6/5$.

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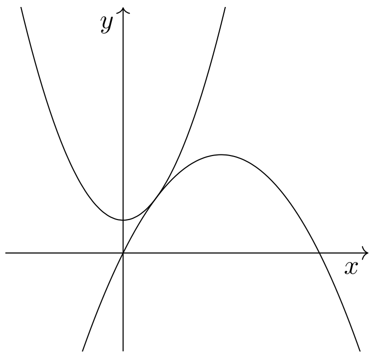
4829. Let A_n be the area of the region of the (x, y) plane defined by the inequality $x^{\frac{1}{2n}} + y^{\frac{1}{2n}} \leq 1$, for $n \in \mathbb{N}$. Prove that

$$\lim_{n\to\infty} A_n = 0.$$

4830. Parabola P_1 is drawn, with equation

$$y = -x(x-2).$$

Parabola P_2 is a transformed version of P_1. It has been enlarged by scale factor $\frac{1}{2}$, and then placed as a positive parabola, with its vertex on the y axis, such that the parabolae are tangent.



Find the y intercept of P_2.

4831. A definite integral is given as

$$I = \int_1^k \log_k x \, dx,$$

where $k > 1$ is a constant.

(a) Find the equation of the tangent to $y = \log_k x$ at $x = 1$.

(b) Show that $y = \log_k x$ is concave for $x > 0$.

(c) Hence, show that $I < \dfrac{(k-1)^2}{2\ln k}$.

4832. A curve is given implicitly by the equation

$$x^2 \sin y + \cos y = x.$$

(a) Show that $\dfrac{dy}{dx} = \dfrac{1 - 2x\sin y}{x^2\cos y - \sin y}$.

(b) Show that there are stationary points where

$$4\sin y\cos y = 1.$$

(c) Hence, show that the curve has infinitely many stationary points.

4833. Four fair six-sided dice are rolled, giving scores X_i for $i = 1, 2, 3, 4$. Show that

$$\mathbb{P}\left(\prod X_i = 64 \,\middle|\, \sum X_i = 12\right) = \tfrac{6}{113}.$$

4834. The parabola $y = x^2$ is reflected in $y = \sqrt{3}x$. Show that the area enclosed by the original parabola and its reflection is $\sqrt{3}$.

4835. A student is attempting to express $3\sin\theta - 4\cos\theta$ in harmonic form. She chooses the form

$$R\cos(\theta - \alpha).$$

Her working is then as follows:

$3\sin\theta - 4\cos\theta \equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$
Equating coeffs, $R\cos\alpha = -4$ and $R\sin\alpha = 3$
Hence, $R = 5$ and $\alpha = \arctan\left(-\frac{3}{4}\right)$
This gives $5\cos\left(\theta - \arctan\left(-\frac{3}{4}\right)\right)$.

The answer is incorrect. Explain the error in her working, and give a corrected version.

4836. Show that the quartic $y = 4x^4 - 4x^3 - 7x^2 + 4x + 3$ has a line of symmetry.

4837. Prove that, for a fixed perimeter P, the pentagon of greatest area is regular.

4838. This question concerns the curve $y = \mathrm{f}(x)$, where f is a function defined over $\mathbb{R}$ by

$$\mathrm{f}(x) = \frac{3\sin x}{3 + 2\cos x}.$$

(a) Show that the closest stationary points to the origin are at $x = \pm\pi \mp \arccos\frac{2}{3}$.

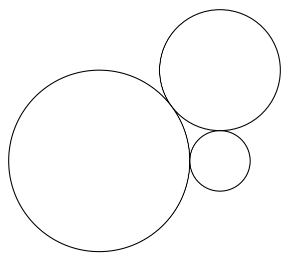
(b) You are given that the first six derivatives, each evaluated at $x = 0$, are

k	1	2	3	4	5	6
$\mathrm{f}^{(k)}(0)$	0.6	0	0.12	0	-0.12	0

Explain the significance, in terms of the shape of the curve, of the fact that $\mathrm{f}^{(k)}(0) \approx 0$ for $k = 2, 3, 4, 5, 6$.

(c) Sketch the curve.

4839. Three circles of radius 1, 2 and 3 are all tangent to each other.



Show that it is possible to draw a rectangle of area $30 + 12\sqrt{6}$ into which they all fit.

4840. A loop of smooth, light string is pulled taut by three forces, forming a triangle. Let the angles in the triangle of string be labelled α, β, γ, and the associated angles in the triangle of forces A, B, C. Prove that

$$\alpha = 90° - \tfrac{1}{2}A$$
$$\beta = 90° - \tfrac{1}{2}B$$
$$\gamma = 90° - \tfrac{1}{2}C.$$

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4841. In this question, y is not identically zero.

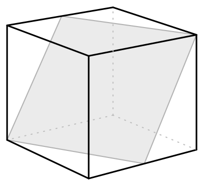
An integral equation is given, for $x \in [0, \pi/2]$, as

$$\int y\,dx = \sqrt{1-y^2} + c.$$

Solve for y in terms of x. Give your answer in a fully simplified form.

4842. A cube has vertices V_i. Two distinct edges V_1V_2 and V_3V_4, which share no endpoints, are chosen at random. Their midpoints are denoted M and N. Points A and B are then chosen at random from among V_5, V_6, V_7, V_8. The quadrilateral $AMBN$ is formed.

In the first trial, $AMBN$ is a rhombus:



Determine the probability that, in a second trial, another such rhombus is formed.

4843. Two curves are given by $x = 4\cos t, y = 6 + 4\sin t$ for $t \in \mathbb{R}$, and $y = \frac{1}{4}x^2$. Find, in exact form, the shortest distance between the two curves.

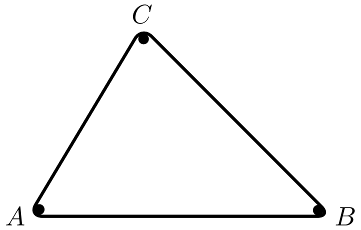
4844. Evaluate $\displaystyle\lim_{L\to\infty} \int_{\frac{1}{2}}^{L} \frac{1}{x^2(x+1)^2}\,dx$ exactly.

4845. Functions $\mathrm{f}, \mathrm{g}, \mathrm{h}$ are defined over $\mathbb{R}$ as

$$\begin{aligned} \mathrm{f}(x) &= x^2, \\ \mathrm{g}(x) &= 2x+1, \\ \mathrm{h}(x) &= x^3 - 3x. \end{aligned}$$

Show that $y = \mathrm{fgh}(x)$ has five stationary points, three of which are on the x axis.

4846. A loop of string is passed, under tension T, around three smooth pegs, which sit at the vertices of an acute triangle ABC.



Let F_A be the resultant force exerted by the string on the peg at A. Show that, in terms of the side lengths of $\triangle ABC$,

$$F_A = \sqrt{\frac{(b+c)^2 - a^2}{bc}}\,T.$$

4847. In this question, do not use any calculus.

Let $\mathrm{L}(a)$, for $a > 1$, be defined as the area of the region enclosed by the curve $y = \frac{1}{x}$ and the lines $y = 0$, $x = 1$ and $x = a$.

(a) Sketch the relevant region.

(b) Show, using a graphical argument, that the area function L obeys the law of logarithms

$$\mathrm{L}(ab) = \mathrm{L}(a) + \mathrm{L}(b).$$

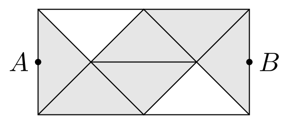
4848. The following DEis a *Bernoulli equation*:

$$\frac{dy}{dx} = \frac{y^2 - xy}{x^2}.$$

Using the substitution $y = ux$, find the general solution, giving y as a simplified function of x.

4849. Sketch $y = \cos^{5.5} x$.

4850. The eight regions of the schematic map below are each shaded, or left blank, at random. The choice for each region is independent. In the example, six contiguous regions have been shaded.



Show that the probability that points A and B end up connected by contiguous shaded regions is $\frac{17}{256}$. (Regions are contiguous if they share a border, but not if they only share a vertex.)

4851. An object of mass 1 kg moves along an x axis, under the action of a resultant force F Newtons. Initially, at $t = 0$ seconds, the particle is at rest at $x = 0$. The force acts in the positive x direction, for the time period $t \in [0, n)$, where $n \in \mathbb{N}$. Its magnitude is given, for $k \in \{1, 2, ..., n\}$, by

$$F = \begin{cases} k, & k-1 \le t < k, \\ 0, & t \ge n. \end{cases}$$

Find, in terms of n,

(a) the velocity at $t = 2n$,

(b) the displacement at $t = 2n$.

You may wish to use the result

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

4852. Determine the value of $\displaystyle\lim_{x\to 1} \frac{x^n - 1}{x^n - x}$, for $n \in \mathbb{N}$.

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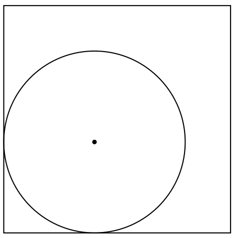
4853. Prove the following result:

$$\int \arcsin x \, dx = x \arcsin x + \sqrt{1-x^2} + c.$$

4854. *Twin primes* are a pair of prime numbers which differ by 2. The first pair is $(3, 5)$. Prove that, for all subsequent pairs of twin primes (a, b), the sum $a + b$ is divisible by 12.

4855. A square of side length 1 has a circle of radius $R \in (1/4, 1/2)$ drawn inside it, tangent to two sides as shown below:



Another circle of radius $r < R$ is to be placed in the square. It must be tangent to the first circle and also to two sides of the square. Show that the radius r must satisfy one of the following:

$$r = \begin{cases} (3-2\sqrt{2})R, \\ \dfrac{2-2R}{1+\sqrt{2}}, \\ \left(1-\sqrt{R}\right)^2. \end{cases}$$

4856. A function is defined, over the reals, as

$$\mathrm{E}(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.$$

Assuming that the sum converges, prove that $\mathrm{E}(x)$ is the series expansion of the exponential function.

4857. A parametric curve is defined, for $t \in \mathbb{R}$, by

$$x = \sin 2t,$$
$$y = \sin t.$$

Show that part of this curve lies outside the ellipse

$$8x^2 + 4y^2 = 9.$$

4858. Sequences $\{u_n\}$ and $\{v_n\}$ are as follows:

- $\{u_n\}$ is arithmetic and $\{v_n\}$ is geometric,
- all terms are non-negative integers,
- the common ratio of $\{v_n\}$ is an integer,
- $u_1 = v_1$,
- $u_3 = v_3$,
- u_2 and v_2 differ by one.

Determine all possible sequences $\{u_n\}$ and $\{v_n\}$.

4859. In this question, assume the following result: if

$$\overrightarrow{OP} = \mu_1 \overrightarrow{OX} + \mu_2 \overrightarrow{OY},$$

where $\mu_1, \mu_2 > 0$ are constants with $\mu_1 + \mu_2 = 1$, then P lies on the line segment XY.

The convex quadrilateral $ABCD$ has vertices with positions $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$. For some constants $\lambda_i > 0$ with $\sum \lambda_i = 1$, point P has position

$$\mathbf{p} = \lambda_1 \mathbf{a} + \lambda_2 \mathbf{b} + \lambda_3 \mathbf{c} + \lambda_4 \mathbf{d}.$$

Prove that P is inside quadrilateral $ABCD$.

4860. Without assuming the derivative of arctan, show that, if $y = \arctan(\cos x)$, then

$$\frac{dy}{dx} = \frac{-\sin x}{1+\cos^2 x}.$$

4861. A cross-section of a cube of side length 1 is taken. Find the greatest possible area of this cross-section if it is

(a) a triangle,

(b) a quadrilateral,

(c) a hexagon.

4862. The following statements, regarding a polynomial function f defined over $\mathbb{R}$, are linked logically by a one-way implication:

① the range of f is $\mathbb{R}$,

② the range of f'' is $\mathbb{R}$.

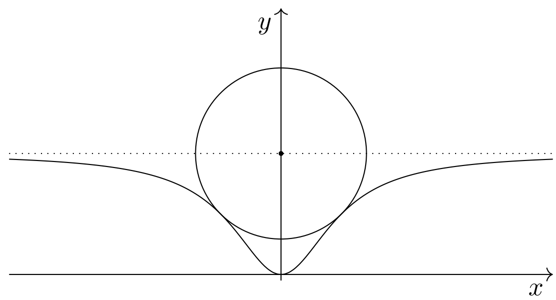
(a) Prove the relevant one-way implication.

(b) Disprove the two-way implication.

4863. The graph below shows a circle centred on the y axis and a curve which is tangent to it, defined by

$$y = \frac{2x^2}{x^2+1}.$$

The curve has a horizontal asymptote, which passes through the centre of the circle.



Determine the radius of the circle.

4864. Show that, for $b \neq 0$,

$$\int_0^b \frac{a^2+x^2}{b^2+x^2} \, dx = b + \frac{\pi\left(a^2-b^2\right)}{4b}.$$

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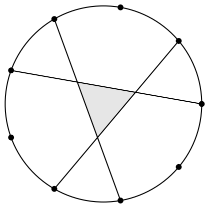
4865. Prove that $y = \cos x$ may be well approximated, for small angles x in radians, by the arc of a circle centred at the origin.

4866. The diagram below depicts nine points arranged symmetrically around the circumference of a unit circle, with some chords added between them.



Show that the side length of the shaded triangle is $2\cos 10°(1 - 4\sin 10°)$.

4867. By considering a sphere as consisting of infinitely many thin discs, use definite integration to prove the formula for the volume of a sphere:

$$V = \tfrac{4}{3}\pi r^2.$$

4868. In a laboratory, a pulley system consists of a light, inextensible string passed over a smooth, movable pulley. The pulley is raised or lowered by machine. Bobs of mass $m_1 < m_2$ are attached to each end of the string. The only forces on the bobs are tension and weight. Find the acceleration a of the string around the pulley if

(a) the pulley does not accelerate,

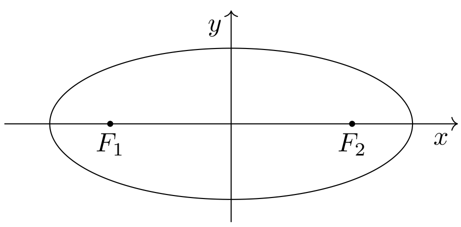
(b) the pulley accelerates upwards at $\frac{1}{2}g$.

4869. An ellipse is given, for $t \in [0, 2\pi)$ by

$$x = a\cos t,$$
$$y = b\sin t.$$

The *foci* F_1 and F_2 of this ellipse are located at

$$F_1, F_2 : \left(\pm\sqrt{a^2 - b^2}, 0\right).$$



(a) Show that, if P is the point with parameter t, then $|F_1P|$ is given by $a + \cos t\sqrt{a^2 - b^2}$.

(b) Find the distance $|F_2P|$.

(c) Hence, prove that that the total distance $|F_1P| + |F_2P|$ is constant.

4870. On a social network, users are either connected to each other or they are not. Prove by contradiction that there must be two users who have precisely the same number of connections.

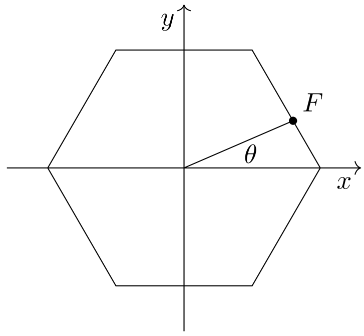
4871. Show that the graph of $|\sin x| = |\sin y|$ consists of infinitely many squares of area $\frac{1}{2}\pi^2$.

4872. A cubic is defined as $y = -\varepsilon x^3 + x^2 + 1$, where the constant ε is small and positive.

(a) Find all stationary points, giving your answer in terms of ε where necessary.

(b) Show that the cubic has exactly one root, which is located at $x \approx 1/\varepsilon$.

4873. A ferret is running at constant speed around the perimeter of a regular hexagon. At the centre of the hexagon, the angle subtended by the position of the ferret is defined as θ. The angular speed ω is defined as the rate of change of θ.



Show that the ratio of the greatest and least values of ω during the motion is $4 : 3$.

4874. Perform the following integral:

$$\int x^3 e^{x^2}\, dx.$$

4875. A circle in the (x, y) plane moves, parametrised by time t, according to

$$(x - 2\sin 2t)^2 + (y - 2\sin t)^2 = 1.$$

Sketch the set of (x, y) points through which the circle passes.

4876. Statement S_k, for $k \in \mathbb{N}$, is given as:

"The product of any seven consecutive natural numbers is necessarily a multiple of k."

Let K be the set of $k \in \mathbb{N}$ for which statement S_k is true. Show that K has 60 elements.

4877. A tangent line is drawn, at $x = a$, to the graph

$$y = x^3 + 3x^2 - 5.$$

Show that the tangent line re-intersects the curve for all values of a except one. This value is to be determined.

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4878. A curve is given, for non-zero constants a, b, by

$$(ax + by) = (bx - ay)^3 .$$

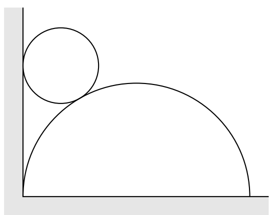
By finding a tangent, or otherwise, show that the origin is a point of inflection.

4879. An iteration I is defined, for some $k \in \mathbb{N}$, by

$$x_{n+1} = \sum_{r=1}^{2k+1} x_n^r.$$

Show that I has exactly two fixed points.

4880. A cylinder and half-cylinder are placed as below, tangent to vertical and horizontal surfaces. Their radii are in the ratio $1 : 3$. They have the same length and are made of the same uniform material. All contacts are smooth.



(a) Show that, at the instant of release from rest, the accelerations are in the ratio $1 : \sqrt{3}$.

(b) Find the initial acceleration of the cylinder.

4881. The arc length L of the graph $y = \mathrm{f}(x)$ between x values p and q is given by the formula

$$L = \int_p^q \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

(a) Using a quarter-circle centred on O, show that the circumference of a circle is given by

$$C = 4r \int_0^r \frac{1}{\sqrt{r^2 - x^2}}\, dx.$$

(b) Hence, verify the formula $C = 2\pi r$.

4882. Three red, three green and three blue counters are placed at random in a row. Given that the red counters have ended up in a single group, find the probability that the blue counters have too.

4883. The relation $(x^2 + y)^2 + y^2 = 1$ defines a curve.

(a) Explain how you know that there are no points on the curve for which

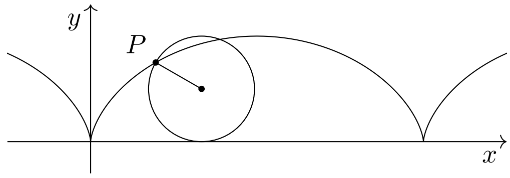
i. $|y| > 1$,

ii. $|x| > 2$.

(b) Find any stationary points.

(c) Hence, sketch the curve.

4884. A wheel of radius 1 rolls along flat ground, without slipping. A point on the wheel is labelled P. This point is at the origin at time $t = 0$. As the wheel rolls, P traces out a curve called a *cycloid*.



(a) Show that the cycloid may be described by

$$x = t - \sin t, \quad y = 1 - \cos t.$$

(b) Show that, over the course of one revolution, the area under the cycloid is three times that of the wheel itself.

4885. The cubic equation $\mathrm{f}(x) = 2x^3 - 8x^2 - 17x - 6 = 0$ has three real roots $a < b < c$, which a student is hoping to find by the Newton-Raphson method. Show that no starting value $x_0 \geq 3$ will converge to b.

4886. Prove the following trigonometric identity:

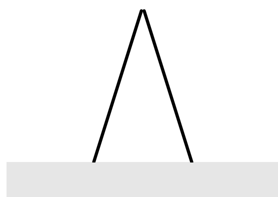
$$(\tan x + \cot x)(\sin x + \cos x) \equiv \sec x + \operatorname{cosec} x.$$

4887. In the following, p and q are prime numbers. Show that the sum S of the integers from 1 to npq which are not divisible by either p or q is

$$S = \tfrac{1}{2}n^2 pq(p-1)(q-1).$$

4888. Two identical playing cards, each of weight W, are leant up against each other. They stand on a high-friction surface, each at angle of inclination $\theta = \arctan \frac{12}{5}$ to the horizontal.

In both parts of this question, the contact between the two cards is modelled such that the reaction forces at that point are horizontal.



(a) Show that the contact force between the two cards has magnitude $\frac{5}{24}W$.

(b) A breeze now blows, exerting a horizontal force of magnitude $\frac{1}{2}W$ at the midpoint of one of the cards. The house withstands the breeze. Show that the coefficient of friction between the cards satisfies $\mu \geq \frac{9}{10}$.

4889. Sequences $\{a_n\}$, $\{b_n\}$ and $\{a_n + b_n\}$ are all GPs. None of their terms is zero. Show that a_n and b_n have the same common ratio.

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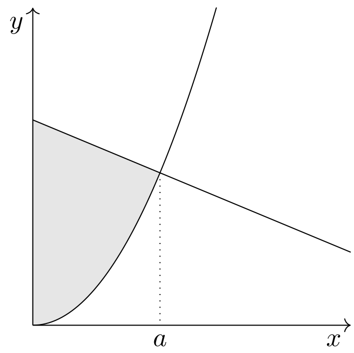
4890. In an archery competition, archers A and B, who are equally skilful, take turns aiming for gold. With each arrow, there are three possibilities:

- gold, with probability p,
- another colour, with probability q,
- failure to score, with probability $1 - p - q$.

An archer wins immediately by hitting gold and loses immediately by failing to score. The archers take turns until one of these outcomes occurs.

The relationship between p and q depends on the distance over which the competition is held. The archers want to choose the distance such that A and B are equally likely to win, irrespective of who goes first. Show that, in this case, $q = 1 - 2p$.

4891. A normal is drawn to the parabola $y = x^2$ at $x = a > 0$. This encloses a region between the the normal, the curve, and the y axis.



The area of the shaded region is 1.452. Find a.

4892. Show that $\arctan x + \arctan {}^1\!/_x \equiv {}^\pi\!/_2$.

4893. Sketch the following graphs on a single set of axes, marking any intercepts.

① $y = x^4 - x^2$,
② $y = x^6 - x^2$.

4894. You are given that

$$\sin 54° = \tfrac{1}{4}\left(1 + \sqrt{5}\right).$$

By considering $(\sin 27° \pm \cos 27°)^2$, show that

$$4 \sin 27° = \sqrt{5 + \sqrt{5}} + \sqrt{3 - \sqrt{5}}.$$

4895. Prove that, for $a, b \in \mathbb{N}$,

$$\int_0^{2\pi} \sin ax \sin bx \, dx = 0.$$

4896. A convex n-gon has vertices with position vectors $\mathbf{a}_1, ..., \mathbf{a}_n$. Prove that every point P in the interior of the n-gon can be expressed, for non-negative constants $\lambda_1, ..., \lambda_n$ with $\sum \lambda_i = 1$, by a position vector of the form

$$\mathbf{p} = \sum_{i=1}^{n} \lambda_i \mathbf{a}_i.$$

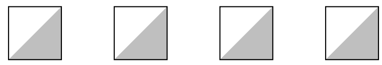
4897. *Faulhaber's formula* states that, with T_n defined as $T_n = \frac{1}{2}n(n+1)$, the sum of fifth powers is

$$\sum_{r=1}^{n} r^5 = \frac{4T_n^3 - T_n^2}{3}.$$

Perform the algebra of a proof, by showing that

$$4T_n^3 - T_n^2 + 3(n+1)^5 = 4T_{n+1}^3 - T_{n+1}^2.$$

4898. The four tiles below are placed together, in random orientations, to form a two-by-two square.

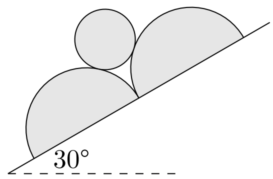


Given that the end result has rotational symmetry, find the probability that its symmetry is order 4.

4899. Using a substitution, determine

$$\int \frac{1}{\sin^3 x - \sin x} \, dx.$$

4900. A smooth cylinder of weight W and radius r rests in equilibrium on top of two half-cylinders, each of radius $2r$. The half-cylinders are attached to a slope, which is inclined at 30° above the horizontal.



(a) Draw a triangle of forces for the cylinder, and show that its interior angles are $\arcsin {}^2\!/_3 \pm 30°$ and $180° - 2\arcsin {}^2\!/_3$.

(b) Hence, show that the larger contact force on the cylinder is given by

$$R_1 = \left(\tfrac{3}{8} + \tfrac{3}{4}\sqrt{\tfrac{3}{5}}\right) W.$$

—— End of 49th Hundred ——

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